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Short Communication

Exact solution of a quadratic nonlinear oscillator

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Abstract

The exact solution of a quadratic nonlinear oscillator (QNO) is presented by using the Jacobian elliptic sine function. The constants in the solution are expressed in terms of the initial conditions. The formulas for the period of oscillation T and amplitude B (for $x \leqslant 0$) are also given. This exact solution can serve as a benchmark solution for various approximate solutions. It also can be utilized for educational purpose.

1. Introduction

Mickens [1] has recently examined the periodic solution of quadratic nonlinear oscillators (QNOs). QNOs give useful models for both the testing of perturbation methods and the analysis of various phenomena in the physical and engineering sciences [1,2]. In general, the quadratic nonlinear oscillating differential equations are special cases of the second-order equation [1]

$$\ddot{x} + x + \alpha x^2 + \beta x\dot{x} + \gamma(\dot{x})^2 = 0, \tag{1}$$

where (α, β, γ) are parameters and overdots denote differentiation with respect to time t. An example of Eq. (1) is [2]

$$\ddot{x} + x + \varepsilon x^2 = 0, \quad x(0) = A > 0, \quad \dot{x}(0) = 0, \tag{2}$$

which is used as a mathematical model of the human eardrum oscillation [2].

In general, nonlinear problems are not amenable to exact treatment and approximate techniques are often resorted to. There are several approximate solutions to Eq. (2). They include the LP method solution [2], the improved harmonic balance method solution [3] and iteration approach solution [4]. Fortunately, Eq. (2) does have the exact solution. Rand [5] presented an exact solution to Eq. (2) for $\varepsilon = 1$, which has the form:

$$x = a_1 + a_2 \mathrm{sn}^2, \tag{3}$$

where $\mathrm{sn} = \mathrm{sn}(\omega t + b, m)$ is the Jacobian elliptic sine function and where a_1, a_2, ω, b and m are constants. In terms of m (m is the square modulus of the function sn), the constants a_1, a_2 and ω can be expressed

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in the form [5]:

$$a_1 = \frac{1+m-\sqrt{\lambda}}{2\sqrt{\lambda}}, \quad a_2 = -\frac{3m}{2\sqrt{\lambda}}, \quad \omega = \frac{1}{2\lambda^{1/4}}, \tag{4a–c}$$

where

$$\lambda = m^2 - m + 1. \tag{5}$$

For free vibrations, Rand's solution is not convenient because the constants (a_1, a_2, ω, m) are not expressed explicitly in terms of the initial conditions. In this paper, we still use Eq. (3) as the exact solution to Eq. (2), but the constants will be expressed explicitly in terms of A.

2. Solution method

Using the initial conditions in Eq. (2), Eq. (3) can be rewritten as

$$x(t) = A + a_2\mathrm{sn}^2 = A + a\,\mathrm{sn}^2(\omega t, m). \tag{6}$$

The first and second time derivatives of Eq. (6) are

$$\dot{x} = 2a\omega\,\mathrm{sn}(\omega t, m)\mathrm{cn}(\omega t, m)\mathrm{dn}(\omega t, m), \tag{7}$$

$$\ddot{x} = 2a\omega^2[1 - 2(1+m)\mathrm{sn}^2 + 3m\,\mathrm{sn}^4]. \tag{8}$$

In Eq. (7), cn and dn are the Jacobian elliptic cosine function and Jacobian elliptic function of the third kind, respectively. Substituting Eqs. (6) and (8) into Eq. (2) gives

$$2a\omega^2 + A + \varepsilon A^2 + [a + 2Aa\varepsilon - 4(1+m)a\omega^2]\mathrm{sn}^2 + (6ma\omega^2 + \varepsilon a^2)\mathrm{sn}^4 = 0. \tag{9}$$

Setting the coefficients of sn^0, sn^2 and sn^4 to zero leads to the following algebraic equations:

$$2a\omega^2 + A + \varepsilon A^2 = 0. \tag{10}$$

$$a + 2Aa\varepsilon - 4(1+m)a\omega^2 = 0, \tag{11}$$

$$6ma\omega^2 + \varepsilon a^2 = 0. \tag{12}$$

From Eq. (12), we have

$$a = \frac{-6m\omega^2}{\varepsilon}. \tag{13}$$

Substituting this equation into Eq. (10) yields

$$12m\omega^4 - \varepsilon A(1+\varepsilon A) = 0. \tag{14}$$

We obtain from Eq. (11)

$$\omega^2 = \frac{1+2\varepsilon A}{4(1+m)}. \tag{15}$$

Substituting Eq. (15) into Eq. (14) and making some arithmetical manipulations gives

$$4p(1+p)m^2 - [3 + 4p(1+p)]m + 4p(1+p) = 0, \quad p = \varepsilon A. \tag{16}$$

Solving for m yields

$$m_{1,2} = \frac{3 + 4p(1+p) \pm \sqrt{9 + 24p(1 - p - 4p^2 - 2p^3)}}{8p(1+p)}. \tag{17}$$

If $\varepsilon \to 0$ ($p \to 0$), we should have $m = 0$ (see Section 3.1.). Therefore, we obtain

$$m(p) = \frac{1}{2} + \frac{3-(1+2p)\sqrt{3(1-2p)(3+2p)}}{8p(1+p)} = \frac{1}{2} + \frac{3(2p^2+2p-1)}{3+(1+2p)\sqrt{3(1-2p)(3+2p)}}. \quad (18)$$

Substituting Eq. (18) into Eq. (15) and simplifying the resulting expression results in

$$\omega(p) = \frac{1}{2}\sqrt{\frac{1}{2} + p + \frac{1}{6}\sqrt{3(1-2p)(3+2p)}}. \quad (19)$$

Substituting Eqs. (18) and (19) into Eq. (13) and making some arithmetical manipulations leads to

$$a(\varepsilon, A) = \left[\sqrt{3(1-2p)(3+2p)} - 3(1+2p)\right]/4\varepsilon = \frac{-12p(1+p)}{\varepsilon\left[\sqrt{3(1-2p)(3+2p)} + 3(1+2p)\right]}. \quad (20)$$

Thus, the exact solution of Eq. (2) is described by Eq. (6) with m, ω and a given by Eqs. (18)–(20) in terms of A.

3. Discussion

It follows from Eq. (18) that we must have $p \leqslant \frac{1}{2}$. But if $p = \frac{1}{2}$, Eq. (2) has a homoclinic orbit with period $+\infty$ [6]. Therefore, the periodic solution given by Eq. (6) is valid for any value of p in the interval [0, 0.5).

3.1. *The special case* $\varepsilon = 0$

If $\varepsilon = 0$, Eq. (2) becomes

$$\ddot{x} + x = 0, \quad x(0) = A > 0 \quad \dot{x}(0) = 0. \quad (21)$$

From Eqs. (18)–(20), we have

$$m(0) = 0, \;\; \omega(0) = \tfrac{1}{2}, \;\; a(0, A) = \lim_{\varepsilon \to 0} a = -2A. \quad (22a\text{–}c)$$

In this case, Eq. (6) has the form

$$x(t) = A - 2A\,\mathrm{sn}^2\left(\frac{t}{2}, 0\right) = A - 2A\sin^2\left(\frac{t}{2}\right) = A\cos t. \quad (23)$$

which is just the solution to Eq. (21).

3.2. *The amplitude* B *for* $x \leqslant 0$

Assuming that the system oscillates between asymmetric limits $[-B, A]$ ($B > 0$), we have from Eq. (6)

$$-B = A + a. \quad (24)$$

Substituting Eq. (20) into Eq. (24) produces

$$B(\varepsilon, A) = \frac{3+2p-\sqrt{3(1-2p)(3+2p)}}{4\varepsilon} = \frac{1}{4\varepsilon}\left[3 + 2\varepsilon A - 3\sqrt{1 - \frac{4}{3}\varepsilon A(1+\varepsilon A)}\right]. \quad (25)$$

Obviously,

$$B(0, A) = \lim_{\varepsilon \to 0} B = A, \;\; B\left(\frac{0.5}{A}, A\right) = 2A. \quad (26a,b)$$

Eq. (25) can be rewritten as

$$B(\varepsilon, A) = \frac{\sqrt{3+2p}[\sqrt{3+2p} - \sqrt{3(1-2p)}]}{4\varepsilon} = \frac{2p\sqrt{3+2p}}{\varepsilon[\sqrt{3+2p} + \sqrt{3(1-2p)}]}. \quad (27)$$

It can be easily shown that

$$\sqrt{3+2p} \geqslant \sqrt{3(1-2p)}, \quad 0 \leqslant p \leqslant 0.5. \tag{28}$$

Therefore, we have

$$A = \frac{2p\sqrt{3+2p}}{\varepsilon[\sqrt{3+2p}+\sqrt{3+2p}]} \leqslant B(\varepsilon, A) \leqslant \frac{2p\sqrt{3+2p}}{\varepsilon\sqrt{3+2p}} = 2A. \tag{29}$$

Obviously, if $p \neq 0$ and 0.5,

$$A < B(\varepsilon, A) < 2A. \tag{30}$$

It is interesting to note that in Ref. [6] Eq. (25) is obtained by solving the cubic algebraic equation

$$\frac{B^2}{2} - \frac{\varepsilon B^3}{3} = \frac{A^2}{2} + \frac{\varepsilon A^3}{3}. \tag{31}$$

3.3. *The exact period of oscillation T*

Using the relation $\mathrm{sn}(u \pm 2F) = -\mathrm{sn}(u)$ [7], we have

$$\mathrm{sn}^2(u \pm 2F) = \mathrm{sn}^2(u), \tag{32}$$

where F is the complete elliptical integral of the first kind given by the following equation [7]:

$$F = F\left(k, \frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}, \quad k^2 = m. \tag{33}$$

Let T denote the exact period of the oscillation. Then from Eq. (6) we obtain

$$a\mathrm{sn}^2(\omega t \pm \omega T, m) = a\mathrm{sn}^2(\omega t, m). \tag{34}$$

A comparison of Eq. (34) with Eq. (32) gives

$$T(p) = \frac{2F}{\omega}. \tag{35}$$

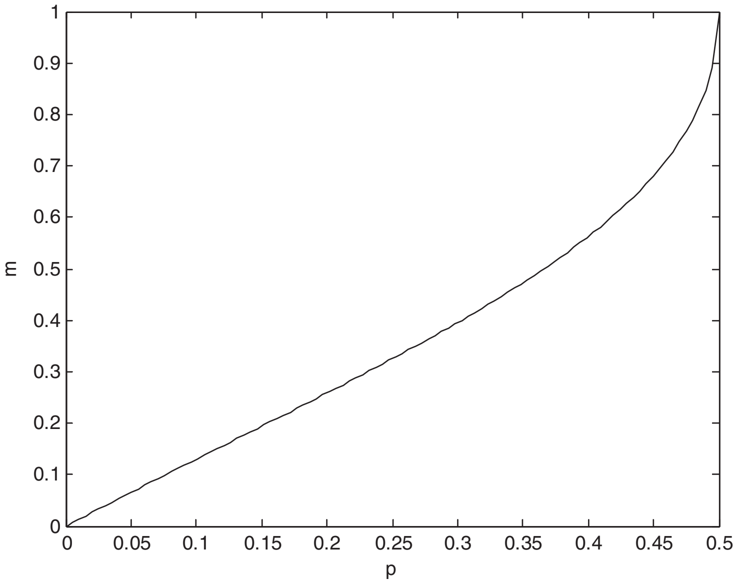

Fig. 1. Variation of m versus p.

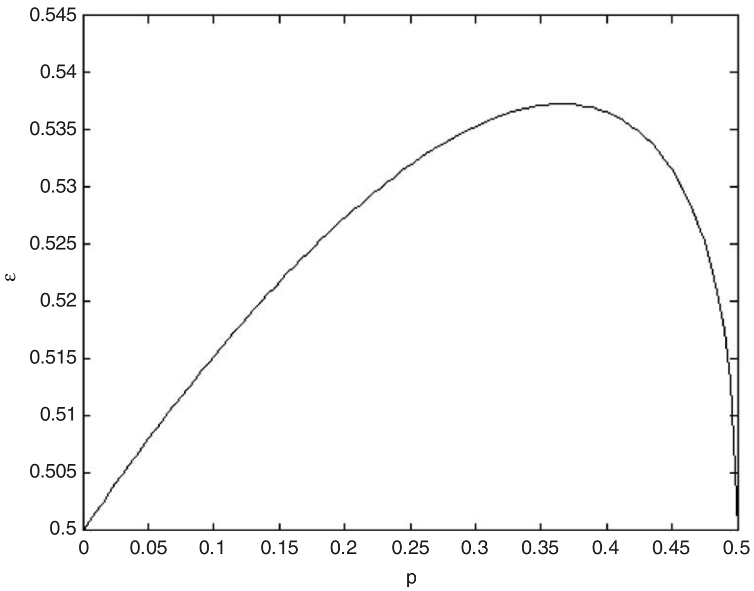

Fig. 2. Variation of ω versus p.

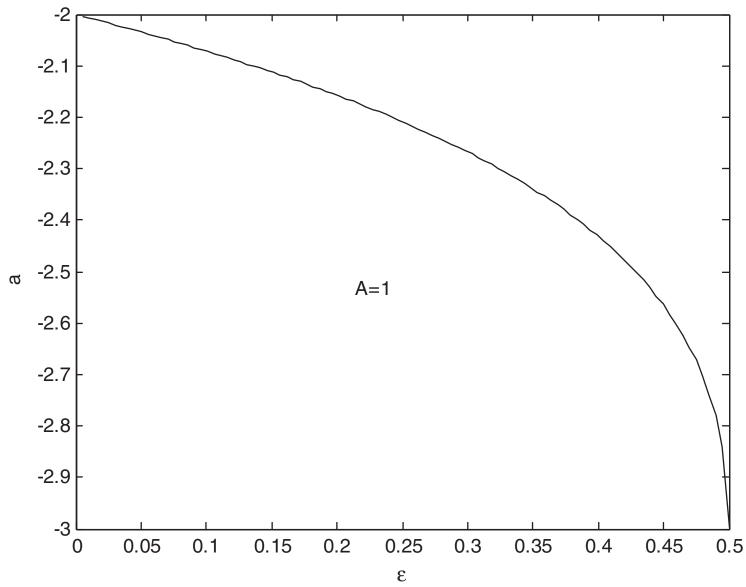

Fig. 3. Variation of a versus ε for $A = 1$.

The two special cases are

$$T(0) = 2\pi, \quad T(0.5) = +\infty. \tag{36a,b}$$

3.4. The graphs of m, ω, a, B, T and $x(t)$

The graphs of m, ω, a, B, T and $x(t)$ are depicted in Figs. 1–5, respectively. Fig. 1 shows that $0 \leqslant m \leqslant 1$ for $0 \leqslant p \leqslant 0.5$. In Figs. 3 and 4, we let $A = 1$. Fig. 5 shows that for p going from 0 to 1, T goes from 2π to infinity. Fig. 6 gives a typical periodic solution $x(t)$ with $A = 1$ and $\varepsilon = 0.45$ for the time in one period.

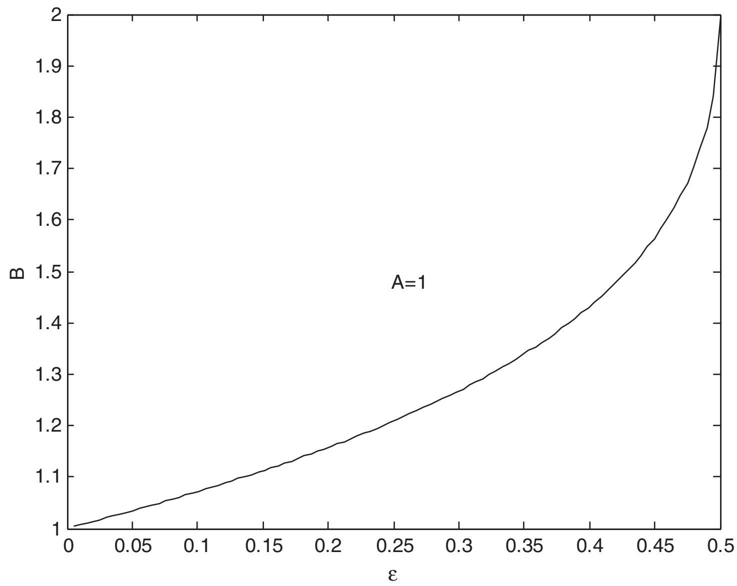

Fig. 4. Variation of B versus ε for $A = 1$.

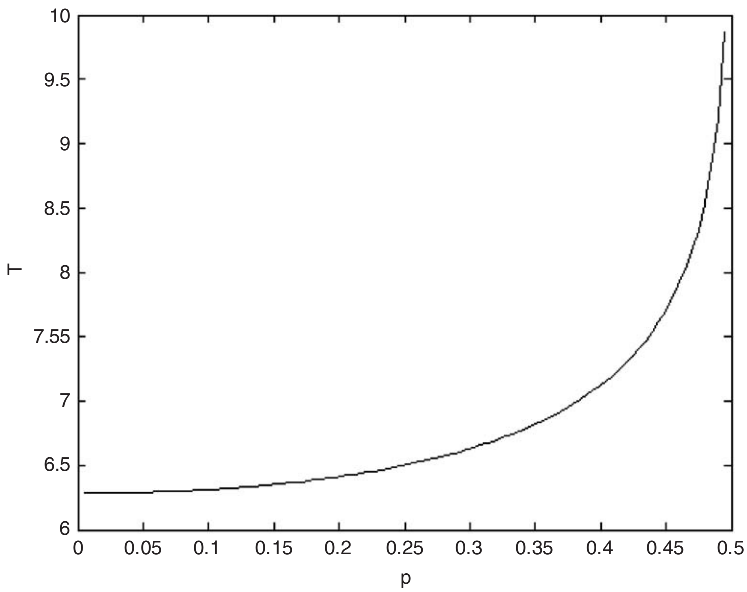

Fig. 5. Variation of T versus p.

4. Conclusions

The exact solution (Eq. (6)) of a QNO modeled by Eq. (2) has been presented. In the solution the constants (a, ω, m) are expressed in terms of the initial conditions. The formulas for the period of oscillation T and amplitude B (for $x \leqslant 0$) have been also derived. The exact solution presented in this paper can serve as a benchmark solution for the purpose of contrasting various approximate solutions with it. It also can be utilized for educational purpose. The work in this paper can be considered as a companion to previous work

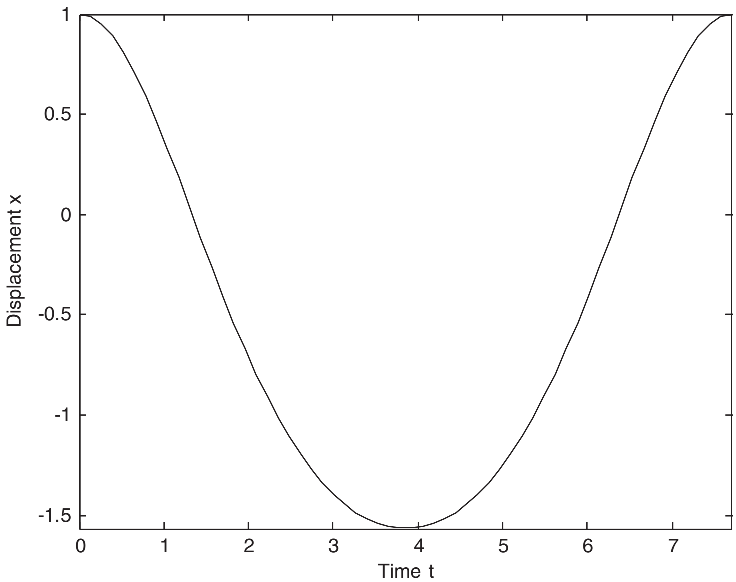

Fig. 6. A typical periodic solution $x(t)$ with $A = 1$ and $\varepsilon = 0.45$ for the time in one period.

on the Duffing equation:

$$\ddot{x} + x + \varepsilon x^3 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{37}$$

For comparison, the exact solution of Eq. (37) [2] is rewritten here:

$$x(t) = A\mathrm{cn}(\omega t, m), \tag{38}$$

where

$$\omega = \sqrt{1 + \varepsilon A^2}, \quad m = k^2 = \frac{\varepsilon A^2}{2(1 + A^2)}. \tag{39a,b}$$

The corresponding exact period of oscillation is given by the expression [2]

$$T = \frac{4F}{\omega} = \frac{4F}{\sqrt{1 + \varepsilon A^2}}. \tag{40}$$

The exact solutions of the QNO (Eq. (2)) and the cubic nonlinear oscillator (Eq. (37)) are expressed by Jacobian elliptic functions. Now we consider the mixed parity differential equation [2]

$$\ddot{x} + x + \alpha x^2 + \beta x^3 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{41}$$

where α and β are constants. Does Eq. (41) have the exact solution which can be described by Jacobian elliptic functions? This is a somewhat difficult problem and needs further research.

Acknowledgements

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